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Some Exact Results in Supersymmetric Theories Based on Exceptional Groups

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Abstract

We begin an investigation of supersymmetric theories based on exceptional groups. The flat directions are most easily parameterized using their correspondence with gauge invariant polynomials. Symmetries and holomorphy tightly constrain the superpotentials, but due to multiple gauge invariants other techniques are needed for their full determination. We give an explicit treatment of G_2 and find gaugino condensation for $N_f \leq 2$, and an instanton generated superpotential for $N_f = 3$. The analogy with $SU(N_c)$ gauge theories continues with modified and unmodified quantum moduli spaces for $N_f = 4$ and $N_f = 5$ respectively, and a non-Abelian Coulomb phase for $N_f \geq 6$. Electric variables suffice to describe this phase over the full range of N_f. The appendix gives a self-contained introduction to G_2 and its invariant tensors.

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1. Introduction

Recent exact results have reinvigorated the study of supersymmetric gauge theories.[1] These results follow from applying the powerful constraints of symmetry and holomorphy of the superpotential. They are interesting for several reasons. First, our understanding of how nature could be described by a theory with spontaneously broken supersymmetry is far from complete. A better understanding of such models with dynamical supersymmetry breaking should yield further insight into this problem. Second, until now the strong-coupling behavior of non-abelian gauge theories has been poorly understood. The exact results offer new approaches to confinement and other interesting features of the vacua of these theories. Furthermore, the growing evidence for strong-weak coupling duality suggests the possibility of opening up a completely new window on strongly coupled theories, through which much more may be learned about their physical properties.

By applying these tools to a variety of theories one hopes to get a better understanding of their underpinnings and of the different phenomena that can occur in supersymmetric gauge theories. The list of models studied has been growing, and includes the simple groups $SU(N)$ [2,3], $SO(N)$ [3,4], and $SP(2N)$ [5], with matter in the fundamental, as well as theories with matter in higher representations [6-12] and product groups[12]. In this paper we initiate a study of the remaining simple groups, the exceptional groups G_2, F_4, E_6, E_7, and E_8.

Beyond our interest in expanding the knowledge of supersymmetric theories and the tools used to study them, there are at least two reasons to consider the exceptional groups. First, the largest exceptional group, E_8, arises in the gauge symmetry $E_8 \times E_8$ of the heterotic string. Furthermore, it has been proposed that non-perturbative effects such as gaugino condensation in one of the E_8 factors[13,14] or the racetrack variants[15] could be the origin of supersymmetry breaking in string theory. This motivates us to better understand such strong-coupling phenomena in E_8 and its subgroups, which include the other exceptional groups. Second, even if string theory does not describe the physical world, E_6 models have been seriously considered as possible grand unified theories. If this group appears in nature, it could also conceivably play a role in the supersymmetry breaking sector.

[1] For a recent review with complete references see [1].

Study of exceptional groups is more difficult than that of the other simple groups due to their greater algebraic complexity. In particular, the problem of explicitly parameterizing flat directions appears formidable. An alternate approach is to use the result that the flat directions correspond to gauge invariant polynomials.[2] One can then use known results about invariant tensors in the exceptional groups to attempt to construct all invariant polynomials, and take these as a starting point for the analysis. Even this problem is difficult; so far we have only managed to explicitly treat G_2. Nonetheless, this or other closely related approaches based on investigating the decomposition of these models under maximal subgroups should in principle yield an exact treatment of the remaining exceptional groups.

We begin the next section by investigating some generic features of exceptional groups. We give a brief discussion of the problem of parameterizing vacua, then use symmetries to find general constraints on the form of the superpotential. Unfortunately it appears that symmetry arguments alone are not sufficient to determine the superpotential. We then investigate the possible emergence of a non-abelian Coulomb phase for certain values of N_f, the number of flavors.

In section two we give a detailed analysis of the group G_2. We are able to deduce the form of the superpotential for $N_f \leq 3$ by using knowledge of the invariant polynomials, symmetry constraints, and the technique of "integrating in[18]." We find gaugino condensation at $N_f \leq 2$ and an instanton generated superpotential at $N_f = 3$. As in the case of $SU(N)$, we also find a modified quantum moduli space for $N_f = 4$ and a quantum moduli space equivalent to the classical one at $N_f = 5$. We then argue that for $N_f \geq 6$ the theory should have a non-abelian Coulomb phase as its infrared description. A minor novelty is that the "electric" description should be valid all the way down to $N_f = 6$: there is no domain where a magnetic description is mandatory to describe the dynamics. This is fortunate, since we have not yet been able to deduce the dual magnetic theory.

The appendix contains a more or less self-contained treatment of G_2. We explicitly construct this group in a way that its maximal $SU(3)$ subgroup is manifest by treating it as the subgroup of $SO(7)$ that leaves a real spinor invariant. From this construction we derive the invariant tensors and the relations among them.

Upon completion of this work, we received [19], which arrives at many of the same results in the G_2 theory.

[2] This result has long been implicit in the literature; recent proofs of it are [16,17].

2. General results in exceptional models

In this section we will make some general observations on models based on exceptional groups, with matter "quark" fields Q^i_α in the defining, or fundamental, representation. Here greek indices are group indices and latin indices label flavors. Dimensions of these representations[20] are shown in table I. With the exception of E_6 the representations are real. We denote quarks in the anti-fundamental of E_6 by $\bar{Q}^\alpha_{\bar{i}}$. In the case of E_8, the adjoint is the smallest representation and will be used for the quarks.

	D_F	D_A	C_F	C_A	Primitive invariants
G_2	7	14	1	4	$\delta^{\alpha\beta}, f^{\alpha\beta\gamma}$
F_4	26	52	3	9	$\delta^{\alpha\beta}, d^{\alpha\beta\gamma}$
E_6	27	78	3	12	$d^{\alpha\beta\gamma}$
E_7	56	133	6	18	$f^{\alpha\beta}, d^{\alpha\beta\gamma\delta}$
E_8	–	248	–	30	$\delta^{AB}, C_3^{ABC}, \cdots$

Table I. Shown are some properties of the exceptional groups. D_F and D_A denote the dimensions of the fundamental and adjoint respectively, and C_F and C_A the Dynkin index of these representations, with normalization convention $C_F(SU(N)) = \frac{1}{2}$.

Study of these theories requires a parameterization of their D-flat directions, that is, solutions of the constraint

$$D^A = \sum_I Q^{*\alpha}_I T^{A\beta}_\alpha Q^I_\beta = 0 \tag{2.1}$$

where $T^{A\beta}_\alpha$ are the group generators. Although explicit parameterizations of these can be given for the non-exceptional groups [21,4,5], the complicated algebraic structure of the exceptional groups makes them more challenging. Alternatives are to decompose the exceptional groups into non-exceptional subgroups, or to use the fact [16,17] that flat directions can be parametrized by the gauge invariant polynomials in the quark fields. In this paper we adopt the latter approach.

To form these polynomials we need the invariant tensors in the fundamental representation. The "primitive" tensors from which these can be constructed are given for groups other than E_8 in [22]. In addition to the tensors δ^α_β and $\epsilon^{\alpha_1\cdots\alpha_{D_F}}$, which are invariant in all cases, the exceptional groups have either fully symmetric primitive tensors, denoted by d's in table I, or totally antisymmetric primitives, denoted by f's in table I.

For E_8, with quarks in the adjoint, the invariant tensors are not explicitly known. Two of them are δ^{AB} and the structure constants c^{ABC}. There are also independent Casimirs at orders 8, 12, 14, 18, 20, 24, and 30, which can be used to form invariants[23].

To find all gauge invariants, we must construct all independent contracted products of these tensors. There are a finite number of independent combinations due to the existence of relations among products of the primitive tensors. Some of these are given in [22], although the complete set of these identities is apparently in general not known.

For example, in E_6 the primitives are δ^α_β, $d^{\alpha\beta\gamma}$, $d_{\alpha\beta\gamma}$, $\epsilon^{\alpha_1\cdots\alpha_{27}}$, and $\epsilon_{\alpha_1\cdots\alpha_{27}}$. From these we can also form invariants[24] such as $d_{\alpha\beta\gamma}d^{\gamma\delta\epsilon}$ and $\epsilon^{\alpha_1\cdots\alpha_{27}}d_{\alpha_{27}\beta\gamma}$. However, nontrivial relations such as $d_{\alpha\beta\gamma}d^{\beta\gamma\epsilon} = 10\delta^\epsilon_\alpha$ and the Springer relation

$$d_{\epsilon\phi\eta}\left(d^{\phi\alpha\beta}d^{\eta\gamma\delta} + d^{\phi\alpha\gamma}d^{\eta\beta\delta} + d^{\phi\alpha\delta}d^{\eta\beta\gamma}\right) = \delta^\alpha_\epsilon d^{\beta\gamma\delta} + \delta^\beta_\epsilon d^{\alpha\gamma\delta} + \delta^\gamma_\epsilon d^{\alpha\beta\delta} + \delta^\delta_\epsilon d^{\alpha\beta\gamma} \qquad (2.2)$$

can be used to reduce many of the higher products. In the appendix we will discuss the analogous problem for G_2 in detail.

Note that, with the exception of G_2, even the one flavor case always has more than one non-trivial invariant, *e.g.* $M = \delta^\alpha_\beta Q_\alpha \bar{Q}^\beta$, $D = d^{\alpha\beta\gamma}Q_\alpha Q_\beta Q_\gamma$, *etc.* in E_6. These invariants parameterize the different subgroups to which the quark vevs may break the original group. For example, the **27** can break E_6 to the distinct maximal subgroups $SO(10)$ and F_4.

Although we will not give a full treatment of the supersymmetric theory for arbitrary exceptional groups here, some general features of these theories can be deduced from symmetries. In the next section we will explicitly treat the group G_2, and in that case fill in more of the details.

With N_f flavors, the non-chiral theories G_2, F_4, E_7, E_8 have the classical symmetries $U(1)_A$,

$$Q^i_\alpha \rightarrow e^{i\phi} Q^i_\alpha, \qquad (2.3)$$

$U(1)_X$,

$$Q^i_\alpha(\theta) \rightarrow Q^i_\alpha(\theta e^{-i\phi}), \qquad (2.4)$$

and $SU(N_f)$. In the E_6 theory, these extend to $U(1)_A$,

$$Q^i_\alpha \to e^{i\phi} Q^i_\alpha, \qquad \bar{Q}^\alpha_{\tilde{i}} \to e^{i\phi} \bar{Q}^\alpha_{\tilde{i}}, \tag{2.5}$$

$U(1)_B$,

$$Q^i_\alpha \to e^{i\phi} Q^i_\alpha, \qquad \bar{Q}^\alpha_{\tilde{i}} \to e^{-i\phi} \bar{Q}^\alpha_{\tilde{i}}, \tag{2.6}$$

and $SU(N_f) \times SU(N_f)$, as well as $U(1)_X$. The non-anomalous R symmetries are given by

$$J^\mu_X - \frac{C_A - n_F C_F}{n_F C_F} J^\mu_A, \tag{2.7}$$

where $n_F = N_f$ for G_2, F_4, E_7, and E_8, and $n_F = 2N_f$ for E_6.

These can be used to constrain the form of the superpotential as with other groups. To see this, first recall that the exact result for the Wilsonian β-function gives [25]

$$e^{-\frac{8\pi^2}{g^2(\mu)} + i\theta} = \left(\frac{\Lambda}{\mu}\right)^{3C_A - n_F C_F}, \tag{2.8}$$

where Λ is the UV cutoff. Therefore we can treat $U(1)_A$ as unbroken if we take Λ to transform as

$$\Lambda^{3C_A - n_F C_F} \to e^{2 i n_F C_F \phi} \Lambda^{3C_A - n_F C_F} \tag{2.9}$$

under (2.3). Using this to constrain the superpotential gives

$$W = f\left(\frac{Q^{2n_F C_F}}{\Lambda^{3C_A - n_F C_F}}\right). \tag{2.10}$$

Demanding that W have charge two under $U(1)_R$ then implies

$$W \sim \frac{\Lambda^{\frac{3C_A - n_F C_F}{C_A - n_F C_F}}}{Q^{2n_F C_F/(C_A - n_F C_F)}}. \tag{2.11}$$

However, this together with the $SU(N_f)$ symmetry is in general not sufficient to uniquely fix the potential since there can be more than one invariant with the correct symmetries. G_2 will furnish an explicit example of this in the next section.

Note that for the superpotential to be instanton generated, it must be proportional to $e^{-8\pi^2/g^2} = \Lambda^{3C_A - n_F C_F}$, i.e.

$$C_A - n_F C_F = 1. \tag{2.12}$$

This is only possible [26] for G_2 with $N_f = 3$. In the other cases where the superpotential is not instanton generated, we should nonetheless be able to deduce its form by relating theories using 1) integrating out/in heavy quarks, which changes the number of flavors, and 2) allowing quarks to get large vevs, which changes the size of the group through the Higgs effect. One might for example be able to use these techniques to relate the E_8 theory without matter to other theories with matter fields in the adjoint [6-12] and thus prove gaugino condensation in E_8 theories.

Parallelling the analysis of the non-exceptional groups, notable theories are those where the superpotential must vanish due to vanishing of the quark R charge,

$$n_F = \frac{C_A}{C_F}, \tag{2.13}$$

and those where asymptotic freedom is lost,

$$n_F = 3\frac{C_A}{C_F}. \tag{2.14}$$

As in other theories, it is natural to conjecture the existence of an interacting non-abelian Coulomb phase for some range of $n_F \leq \frac{3C_A}{C_F}$. The exact 1PI β-function is [25]

$$\beta_g = -\frac{g^3}{16\pi^2}\frac{3C_A - (1-\gamma)C_F n_F}{1 - C_A \frac{g^2}{8\pi^2}} \tag{2.15}$$

where

$$\gamma = -C_F \; \frac{D_A}{D_F} \; \frac{g^2}{4\pi^2} \quad + \cdots \tag{2.16}$$

is the anomalous mass dimension, and near $n_F = \frac{3C_A}{C_F}$ cancellation of one-and two-loop terms appears possible. If there is such a fixed point, described by a superconformal theory, as argued in [3] the chiral operators should have dimensions

$$D = \frac{3}{2}|R|, \tag{2.17}$$

where R is the R charge of the operator. Gauge invariant meson operators can be formed in all of the exceptional models, and by (2.17) they have dimension

$$D(QQ) = 3\frac{n_F C_F - C_A}{n_F C_F}. \tag{2.18}$$

This value agrees with that given by $\gamma + 2$, if one assumes vanishing of the β-function (2.15).

The unitarity constraint $D(QQ) \geq 1$ tells us where this hypothesized "electric" description must fail, below

$$n_F = \frac{3}{2}\ \frac{C_A}{C_F}. \tag{2.19}$$

At this value the meson is a free field.

In non-exceptional theories, in a range below this N_f magnetic variables are conjectured [3] to be necessary to describe the IR dynamics. However, the analogous range may not necessarily exist for exceptional groups. In the next section, we'll see that for G_2 a description in terms of composite mesons and baryons appears appropriate at $N_f = 5$; $N_f = 6$ is the direct analogue of $N_f = N_c + 2$ in $SU(N_c)$, and should give the lower bound on the magnetic description. However

$$\frac{3}{2}\ \frac{C_A}{C_F} = 6 \tag{2.20}$$

so $N_f = 6$ also corresponds to the lower bound on the electric description. There is no range where magnetic variables furnish the only possible IR description.

In the other exceptional groups, it is natural to conjecture that the analogue of $N_f = N_c$ in $SU(N_c)$ is

$$n_F = \frac{C_A}{C_F}; \tag{2.21}$$

here the superpotential must vanish and one anticipates a quantum moduli space different from the classical one. (We will show this for G_2.) Taking the potentially dangerous step of pushing the analogy further, in each case the lower bound or the magnetic description would then be at

$$\begin{aligned} n_F &= \frac{C_A}{C_F} + 2 \quad (G_2, F_4, E_7, E_8)\ , \\ n_F &= \frac{C_A}{C_F} + 4 \quad (E_6)\ , \end{aligned} \tag{2.22}$$

and this exceeds (2.19) for all exceptional groups. Thus an electric description may be sufficient in each case.

Nonetheless, a magnetic description could add useful insight into the dynamics of these theories. Even for G_2 we have not yet found this description. Some clues exist; for example in all simple non-exceptional groups the theory is self dual at the value of N_f for which an added adjoint matter field yields a vanishing β-function. For exceptional groups, this would happen at

$$n_F = \frac{2C_A}{C_F}. \tag{2.23}$$

We will make further comments on the G_2 case in the following section.

3. Supersymmetric G_2 gauge theory

The example we consider is an N=1 supersymmetric G_2 gauge theory with N_f flavors of quarks in the fundamental **7** representation. Using the techniques developed in [2],[27] we obtain exact results in the quantum theory. We recover features which are similar to those obtained for $SU(N_c), SO(N_c)$ and $Sp(N_c)$ gauge theories with matter in the fundamental representation. This is further evidence of a set of properties generic to N=1 supersymmetric gauge theories.

In particular we find gluino condensation and instanton generated superpotentials for $N_f \leq 2$ and $N_f = 3$ respectively, with no ground state in the massless limit. For $N_f = 4$ there is a moduli space of inequivalent vacua which is smoothed out in the quantum theory by a one instanton effect. For $N_f = 5$ the classical and quantum moduli space are the same and there is confinement without chiral symmetry breaking at the origin. For $N_f \geq 6$, we expect a nonabelian Coulomb phase to describe the infrared physics.

3.1. Gauge invariant fields

As discussed in the preceding section, the light degrees of freedom on the moduli space can be labeled by the G_2 gauge invariant polynomials of the fundamental quarks subject to possible constraints. As shown in the appendix, these polynomials can be constructed from the three independent invariant tensors $\delta^{\alpha\beta}$, $f^{\alpha\beta\gamma}$ and $\tilde{f}^{\alpha\beta\gamma\delta}$. Combining these with the quarks Q^i_α leads to the composite fields

$$\begin{aligned} M^{ij} &= \delta^{\alpha\beta} Q^i_\alpha Q^i_\beta \\ B_{i_4\cdots i_{N_f}} &= \frac{1}{3!}\epsilon_{i_1 i_2 i_3 i_4 \cdots i_{N_f}} f^{\alpha\beta\gamma} Q^{i_1}_\alpha Q^{i_2}_\beta Q^{i_3}_\gamma \\ F_{i_5\cdots i_{N_f}} &= \frac{1}{4!}\epsilon_{i_1 i_2 i_3 i_4 i_5 \cdots i_{N_f}} \tilde{f}^{\alpha\beta\gamma\delta} Q^{i_1}_\alpha Q^{i_2}_\beta Q^{i_3}_\gamma Q^{i_4}_\delta \ . \end{aligned} \tag{3.1}$$

3.2. Gaugino condensation for $N_f \leq 2$

Due to the antisymmetry of $f^{\alpha\beta\gamma}$ and $\tilde{f}^{\alpha\beta\gamma\delta}$, B and F vanish for $N_f \leq 2$ and the only light fields are the M^{ij}. In addition to holomorphy, $U(1)_R$ symmetry and dimensional analysis, which constrain the form of the superpotential to (2.11), invariance under the $SU(N_f)$ global flavor symmetry further restricts it to be

$$W_{\text{eff}} = \frac{(4-N_f)(\Lambda_{N_f})^{\frac{12-N_f}{4-N_f}}}{\det M^{\frac{1}{4-N_f}}} \,. \tag{3.2}$$

This is exact; the normalization of Λ_{N_f} has been adjusted to set threshold corrections to unity [27]. G_2 has a maximal $SU(3)$ subgroup under which $\mathbf{7} \rightarrow \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1}$ (see appendix). Therefore as we turn on vevs, the generic breaking sequence is

$$G_2 \overset{N_f=1}{\longrightarrow} SU(3) \overset{N_f=2}{\longrightarrow} SU(2) \overset{N_f=3}{\longrightarrow} \phi. \tag{3.3}$$

For $N_f = 1$ or 2, gluino condensation will occur [2] in the sector with the unbroken pure $SU(4-N_f)$ gauge theory through nonzero expectation values of the gluino bilinears,

$$\langle \lambda\lambda \rangle_{4-N_f} \sim \Lambda^3{}_{SU(4-N_f)} \,. \tag{3.4}$$

Matching the couplings at the G_2 breaking scale $v = (\det M)^{1/2N_f}$ relates the scales of the two theories,

$$\Lambda^3{}_{SU(4-N_f)} = v^3 \left(\frac{\Lambda_{N_f}}{v} \right)^{\frac{12-N_f}{4-N_f}} \,. \tag{3.5}$$

Since $\langle \lambda\lambda \rangle_{4-N_f}$ is the coefficient of the dynamically generated F-term [21] we see that gluino condensation leads to a term of the same form as (3.2). Therefore supersymmetry is spontaneously broken but there is no ground state, as in $SU(N_c)$ with $N_f < N_c - 1$.

Adding a tree level mass term $W_{\text{tree}} = m_{ij} M^{ij}$ to (3.2) and solving for M^{ij} gives the expectation values

$$\begin{aligned} \langle M^{ij} \rangle &= \left(\Lambda_{N_f} \right)^{\frac{12-N_f}{4}} (\det m)^{\frac{1}{4}} (\frac{1}{m})^{ij} \\ \det \langle M^{ij} \rangle &= \left(\Lambda_{N_f} \right)^{\frac{(12-N_f)N_f}{4}} (\det m)^{\frac{N_f-4}{4}} \,. \end{aligned} \tag{3.6}$$

The symmetries ensure that (3.6) generalizes to higher values of N_f when $B = F = 0$. For $m^{ij} \neq 0$ there are four different supersymmetric ground states. Notice[26] that this disagrees with the Born-Oppenheimer calculation of the Witten index which would give three.

3.3. Instanton generated superpotential for $N_f = 3$

For $N_f = 3$ the light fields are labeled by the "mesons" $M^{ij} = \delta^{\alpha\beta} Q^i_\alpha Q^j_\beta$ and a "baryon" $B = \frac{1}{3!}\epsilon_{ijk} f^{\alpha\beta\gamma} Q^i_\alpha Q^j_\beta Q^k_\gamma$. Holomorphy, symmetries and dimensions restrict the superpotential to the form

$$W_{\rm eff} = \frac{\Lambda_3^9}{\det M} f\left(\frac{B^2}{\det M}\right). \tag{3.7}$$

Since $W_{\rm eff}$ has the right quantum numbers we expect that it is instanton generated along the flat directions where the gauge symmetry is completely broken. However it seems that the form of $f(x)$ cannot be deduced purely from symmetry arguments. The exact form can instead be found using the "integrating in" technique of [18] by taking G_2 with $N_f = 3$ as the "upstairs" theory and $N_f = 2$ as the "downstairs" theory. The dynamically generated superpotential (3.2) of the downstairs theory is

$$W_d = \frac{2\Lambda_2^5}{\sqrt{\det M_d}} \ . \tag{3.8}$$

Since the upstairs theory contains a non-quadratic gauge invariant there may be additional terms in the downstairs theory of the form $W_{{\rm tree},d} + W_\Delta$. By turning on a tree level superpotential for the "new" flavor Q in the upstairs theory

$$W_{{\rm tree},u} = mQ^3 {\cdot} Q^3 + bB \tag{3.9}$$

and integrating out the heavy quark we find

$$W_{{\rm tree},d} = -\frac{b^2}{4m} \det M_d \ , \tag{3.10}$$

where M_d are the mesons constructed from the remaining quarks. W_Δ is determined by symmetries to be of the form

$$W_\Delta = \frac{b^2}{m} \det M_d \ g\left(\frac{b^2 (\det M_d)^{\frac{3}{2}}}{m\Lambda_2^5}\right) \tag{3.11}$$

and the limits $W_\Delta \to 0$ for $m \to \infty$ and $\Lambda_2 \to 0$ restrict $g(x)$ to be exactly zero. Λ_2 is related to Λ_3 by

$$(\Lambda_2)^5 = \sqrt{E}(\Lambda_3)^{\frac{9}{2}} \tag{3.12}$$

where E is the scale where the couplings match. Symmetries restrict E to have the form

$$E = mh\left(\frac{b^2 \mathrm{det} M_d{}^{\frac{3}{2}}}{m\Lambda_2^5}\right) \tag{3.13}$$

and the limits $E \to m$ as $\mathrm{det}M \to \infty$ and $m \to \infty$ imply that $h(x) = 1$. Therefore the exact matching condition is

$$(\Lambda_2)^5 = \sqrt{m}(\Lambda_3)^{\frac{9}{2}}. \tag{3.14}$$

By combining results (3.8), (3.10), and (3.14), W_u can be obtained from

$$W_n = \frac{2\sqrt{m}\Lambda_3^{\frac{9}{2}}}{\sqrt{detM_d}} - \frac{b^2}{4m}\mathrm{det}M_d - mQ_3 \cdot Q_3 - bB \tag{3.15}$$

by treating m and b as fields and integrating them out. The result is that

$$W_u = \frac{\Lambda_3^9}{\mathrm{det}M - B^2} \tag{3.16}$$

is the dynamically generated superpotential for $N_f = 3$. This theory has no ground state and is similar to $SU(N_c)$ with $N_f = N_c - 1$.

The singularity at $\mathrm{det}M = B^2$ is due to extra massless gluinos at points where some of the gauge symmetry is unbroken. Although the generic breaking sequence was given in (3.3), at $N_f = 3$ there are also non-trivial flat directions which leave an $SU(2)$ subgroup unbroken. In the basis of the appendix these are easily seen to be

$$\begin{aligned} \langle Q^1_\alpha \rangle &= v_1 \delta_{\alpha 7} \\ \langle Q^2_\alpha \rangle &= v_2(\delta_{\alpha 1} + \delta_{\alpha 4}) \\ \langle Q^3_\alpha \rangle &= v_3(\delta_{\alpha 1} - \delta_{\alpha 4}). \end{aligned} \tag{3.17}$$

Along these flat directions one can easily check $\mathrm{det}M = B^2$ is satisfied.

3.4. Modified quantum moduli space for $N_f = 4$

For $N_f = 4$ the light degrees of freedom are labeled by

$$\begin{aligned} M^{ij} &= \delta^{\alpha\beta} Q^i_\alpha Q^j_\beta \\ B_i &= \frac{1}{3!}\epsilon_{ijk\ell} f^{\alpha\beta\gamma} Q^j_\alpha Q^k_\beta Q^\ell_\gamma \\ F &= \frac{1}{4!}\epsilon_{ijk\ell} \tilde{f}^{\alpha\beta\gamma\delta} Q^i_\alpha Q^j_\beta Q^k_\gamma Q^\ell_\delta \ . \end{aligned} \tag{3.18}$$

The classical constraint is

$$\det M - F^2 - B_i M^{ij} B_j = 0 \tag{3.19}$$

which can be seen as a consequence of (A.20) and the Bose symmetry of the quark fields. The expectation value $\det M = \Lambda_4^8$ from (3.6) implies that the classical constraint is modified quantum mechanically to

$$\det M - F^2 - B_i M^{ij} B_j = \Lambda_4^8 \tag{3.20}$$

and the singularities are smoothed out by a one instanton effect. The symmetries do not allow a dynamically generated superpotential, hence there is a moduli space of inequivalent vacua defined by (3.20) and this is different from the classical moduli space. This is similar to $SU(N_c)$ with $N_f = N_c$ flavors.

These results can be independently derived by taking the $N_f = 3$ case as the downstairs theory and integrating in a new flavor. From (3.16) we have the dynamically generated superpotential of the downstairs theory

$$W_d = \frac{\Lambda_3^9}{\det M_d - B^2}. \tag{3.21}$$

Turning on the tree level superpotential

$$W_{\mathrm{tree},u} = mQ^4{\cdot}Q^4 + 2m_I Q^I{\cdot}Q^4 + b^I B_I + fF \tag{3.22}$$

and integrating out the massive quark gives

$$\begin{aligned} W_{\mathrm{tree},d} = -\frac{1}{4m}\Big[&\det M_d b^I (M_d^{-1})_{IJ} b^J + f^2(\det M_d - B^2) + 4b^I m_I B \\ &+ 4m_I (M_d)^{IJ} m_J\Big] \end{aligned} \tag{3.23}$$

for the downstairs theory (I,J = 1,...,3). Taking $W_\Delta = 0$ and the matching condition $(\Lambda_3)^9 = m(\Lambda_4)^8$ and performing the inverse Legendre transformation on the full superpotential of the downstairs theory leads to

$$W_n = \frac{m\Lambda_4^8}{\det M_d - B^2} + W_{\mathrm{tree},d} - W_{\mathrm{tree},u}\ . \tag{3.24}$$

Integrating out m, m_I, b^I, and f from W_n gives $W_u = 0$ for the dynamically generated superpotential as expected. In addition the equations of motion $\frac{\partial W_n}{\partial m} = \frac{\partial W_n}{\partial m_I} = \frac{\partial W_n}{\partial b^I} = \frac{\partial W_n}{\partial f} = 0$ lead to the quantum constraint (3.20).

The $N_f = 4$ theory can be described by the effective superpotential

$$W_{\rm eff} = X(\det M - F^2 - B_i M^{ij} B_j - \Lambda_4^8) \tag{3.25}$$

where X is a Lagrange multiplier field. Perturbing (3.25) by adding a tree level mass term $W_t = m_{ij} M^{ij}$, the results for $N_f < 4$, (3.16), (3.2) can recovered by integrating out one or more massive quarks.

3.5. Quantum moduli space for $N_f = 5$

The light fields and their transformations under the global $SU(5)\times U(1)_R$ flavor symmetry are

$$\begin{aligned} M^{ij} &= \delta^{\alpha\beta} Q^i_\alpha Q^j_\beta && : 15_{\frac{2}{5}} \\ B_{ij} &= \frac{1}{3!}\epsilon_{ijk\ell m} f^{\alpha\beta\gamma} Q^k_\alpha Q^\ell_\beta Q^m_\gamma && : \overline{10}_{\frac{3}{5}} \\ F_i &= \frac{1}{4!}\epsilon_{ijk\ell m} \tilde{f}^{\alpha\beta\gamma\delta} Q^j_\alpha Q^k_\beta Q^\ell_\gamma Q^m_\delta && : \bar{5}_{\frac{4}{5}} \ . \end{aligned} \tag{3.26}$$

The classical constraint which follows from (A.20) is

$$F_i F_j - B_{ik} M^{k\ell} B_{\ell j} - \det M (M^{-1})_{ij} = 0. \tag{3.27}$$

The expectation values (3.6) imply the quantum-mechanical modification of this to

$$F_i F_j - B_{ik} M^{k\ell} B_{\ell j} - \det M (M^{-1})_{ij} = (\Lambda_5)^7 m_{ij}. \tag{3.28}$$

The classical constraints are satisfied in the $m \to 0$ limit, therefore the moduli space of the massless quantum theory is that same as the classical theory.

At the origin it appears that the $SU(5)\times U(1)_R$ chiral symmetry remains unbroken and that all the components of M^{ij}, B_{ij} and F_i are massless. The 't Hooft anomaly matching conditions between the fundamental fermion fields (quarks which transform like $7 \times \mathbf{5}_{-\frac{4}{5}}$ and 14 gluinos) and those of the massless spectrum (3.26) are satisfied

$$
\begin{array}{ll}
SU(5)^3 & 7d^{(3)}(5) = d^{(3)}(15) + d^{(3)}(\overline{10}) + d^{(3)}(\bar{5}) \\
SU(5)^2U(1)_R & 7(-\frac{4}{5})d^{(2)}(5) = (-\frac{3}{5})d^{(2)}(15) + (-\frac{2}{5})d^{(2)}(\overline{10}) + (-\frac{1}{5})d^{(2)}(\bar{5}) \\
U(1)_R^3 & 35(-\frac{4}{5})^3 + 14 = 15(-\frac{3}{5})^3 + 10(-\frac{2}{5})^3 + 5(-\frac{1}{5})^3 \\
U(1)_R & 35(-\frac{4}{5}) + 14 = 15(-\frac{3}{5}) + 10(-\frac{2}{5}) + 5(-\frac{1}{5}).
\end{array}
\tag{3.29}
$$

There is confinement without chiral symmetry breaking at the origin. This is similar to $SU(N_c)$ with $N_f = N_c + 1$ flavors.

A low energy effective superpotential which obeys all the symmetries in the problem is given by

$$
W_{\text{eff}} = \frac{1}{\Lambda_5^7}[F_i M^{ij} F_j - \frac{1}{2} B_{ij} M^{jk} B_{k\ell} M^{\ell i} - \text{det} M - \frac{1}{4}\epsilon^{ijk\ell m} B_{ij} B_{k\ell} F_m] \tag{3.30}
$$

where the constant coefficients have been chosen so that the constraint (3.27) arises from the equation of motion $\frac{\partial W_{\text{eff}}}{\partial M^{ij}} = 0$. In addition, two other constraints are obtained from

$$
\frac{\partial W_{\text{eff}}}{\partial F_m} = 2M^{mi} F_i - \frac{1}{4}\epsilon^{ijk\ell m} B_{ij} B_{k\ell} = 0 \tag{3.31}
$$

and

$$
\frac{\partial W_{eff}}{\partial B_{mn}} = M^{mi} B_{ij} M^{jn} - \frac{1}{2}\epsilon^{mnijk} B_{ij} F_k = 0. \tag{3.32}
$$

These can also be derived from the identities in the appendix (and fix the coefficient in the last term of (3.30)). By adding a tree level superpotential $W_t = m_{ij} M^{ij}$ and integrating out the massive fields we find the quantum constraint (3.28) as the equation of motion. For the special case $W_t = m_{55} M_{55}$, by integrating out the one massive field we recover the results (3.20), (3.25) of the $N_f = 4$ theory from the equations of motion. Similarly the results for $N_f < 4$ can be recovered by giving masses to more of the fields.

3.6. Non-abelian Coulomb phase for $N_f \geq 6$

The analysis of G_2 has closely paralleled that of $SU(N_c)$, so it's not unreasonable to expect that for $N_f \geq 6$, one finds a non-abelian Coulomb phase analogous to that for $N_f \geq N_c + 2$ in $SU(N_c)$. Indeed, the arguments of [3] show that such a phase should exist for $N_f \geq 8$. To see this, note that allowing two of the flavors to get vevs breaks G_2 to $SU(2)$, with $N_f - 2$ remaining flavors. This theory is not asymptotically free for $(N_f - 2) \geq 6$, implying the existence of a non-Abelian Coulomb phase.

As suggested in section one, it is not unreasonable to expect this non-Abelian electric phase to extend down to $N_f = 6$, where the meson field would have dimension one and become free, as in $SU(N_c)$.

Notice that the coincidence between the lower bound on the electric description and the lower bound on the non-Abelian Coulomb phase means that a dual magnetic theory is not necessary to describe the dynamics. However, the ubiquity of such theories suggests one should be sought here as well.

A standard procedure is to identify the duals of the baryons with the baryons of the dual theory. An added complication here is the existence of the two types of baryons, B and F. The R-charge assignments for dual quarks appear to be simplest if F is taken to correspond to the baryon in the dual theory, but one is then faced with identifying B in the dual. The difficulty of interpreting this as a fundamental field suggests either a more complicated group structure or the necessity for fields transforming in different representations. Indeed, the fact that G_2 can be gotten from $SO(7)$ through breaking by a spinor $\mathbf{8}$ vev suggests that a promising route is to investigate the dual of $SO(N)$ theories with both fundamentals and spinors. A reasonable conjecture is that the dual theories are given by $SO(N)$'s with both fundamental and spinor fields.

4. Conclusion

Using the result that flat directions are parametrized by gauge-invariant polynomials, one may extend the exact treatment of supersymmetric gauge theories to the exceptional groups. Symmetries and holomorphy provide stringent constraints on the superpotential, but are not sufficient to fully determine it as there are multiple invariants that can be formed with the correct transformation properties. Nonetheless, other techniques such as "integrating in" can be used to obtain the superpotential. A necessary first step is to determine the algebraically independent gauge-invariant polynomials, and this requires knowledge of the group's invariant tensors and of the relations among them.

This approach has been explicitly used for the group G_2. Gluino condensation was found for $N_f \leq 2$, and an instanton generated superpotential for $N_f = 3$. At higher N_f the theory also parallels the $SU(N_c)$ case: there is a modified quantum moduli space at

$N_f = 4$, a moduli space equivalent to the classical one at $N_f = 5$, and apparently a non-abelian Coulomb phase for $N_f \geq 6$. The dual magnetic description has not been found, but an electric description suffices to treat the full range of N_f.

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Appendix A.

In this Appendix we will give a self contained derivation of features of G_2 that are needed in the main body of the text. A useful reference is [28].

A.1. Construction of G_2

G_2 can be obtained from $SO(7)$ as the subgroup leaving a real spinor, the **8**, invariant. We will use this fact to give an explicit construction of G_2 that also manifests the maximal $SU(3)$ subgroup.

We begin with a Majorana representation for the Dirac matrices of $SO(7)$; one explicit choice of real matrices is

$$\Gamma^1 = \epsilon \otimes \epsilon \otimes \epsilon,\ \Gamma^2 = 1 \otimes \sigma_1 \otimes \epsilon,\ \Gamma^3 = 1 \otimes \sigma_3 \otimes \epsilon,$$

$$\Gamma^4 = \sigma_1 \otimes \epsilon \otimes 1,\ \Gamma^5 = \sigma_3 \otimes \epsilon \otimes 1,\ \Gamma^6 = \epsilon \otimes 1 \otimes \sigma_1,\ \Gamma^7 = \epsilon \otimes 1 \otimes \sigma_3. \tag{47}$$

Here σ_i are the usual Pauli matrices and $\epsilon = i\sigma_2$. It is most convenient to work with a complex basis $\gamma^a, \gamma^{\bar{a}}$ for six of the seven matrices, *e.g.*

$$\gamma^1 = \frac{i\Gamma^1 - \Gamma^4}{2}\ ,\ \gamma^{\bar{1}} = \frac{i\Gamma^1 + \Gamma^4}{2}\ , \text{etc.}$$

and to rename $\bar{\gamma} = \Gamma^7$.

In such a basis, the Dirac algebra $\{\Gamma^I, \Gamma^J\} = -2\delta^{IJ}$ becomes

$$\{\gamma^a, \gamma^{\bar{b}}\} = \delta^{a\bar{b}}\ ,\ \{\bar{\gamma}, \gamma^a\} = \{\bar{\gamma}, \gamma^{\bar{a}}\} = 0\ ,\ \bar{\gamma}^2 = -1\ . \tag{A.1}$$

Note also that $\gamma^{a*} = -\gamma^{\bar{a}}, \gamma^{a\dagger} = \gamma^{\bar{a}}$. The γ^a's and $\gamma^{\bar{a}}$'s behave like fermion creation and annihilation operators, and we can always find a spinor ζ satisfying

$$\gamma^a \zeta = 0, \tag{A.2}$$

corresponding to the Fock vacuum. This also implies

$$i\bar{\gamma}\zeta = \zeta \ ; \tag{A.3}$$

$i\bar{\gamma}$ is analogous to $(-1)^F$.

G_2 is the little group leaving a real spinor η invariant. This subgroup is most easily investigated by rotating η into the form

$$\eta = \zeta + \zeta^* + \zeta^c + \zeta^{c*}, \tag{A.4}$$

where

$$\zeta^c = \gamma^{\bar{1}}\gamma^{\bar{2}}\gamma^{\bar{3}}\zeta. \tag{A.5}$$

Note that

$$\gamma^{\bar{a}}\zeta^c = 0. \tag{A.6}$$

The generators of $SO(7)$ are the 21 real combinations of

$$\gamma^{ab} \ , \ \gamma^{\bar{a}\bar{b}} \ , \ \bar{\gamma}\gamma^a \ , \ \bar{\gamma}\gamma^{\bar{a}} \ , \quad \text{and} \quad \gamma^{a\bar{b}}. \tag{A.7}$$

The six generators $\gamma^{a\bar{b}}, a{\neq}b$, automatically annihilate η. One can also easily show that the two generators

$$\gamma^{1\bar{1}} - \gamma^{2\bar{2}}, \gamma^{2\bar{2}} - \gamma^{3\bar{3}} \tag{A.8}$$

annihilate η, as do the six generators

$$\bar{\gamma}\gamma^a + \frac{i}{2}\epsilon^{abc}\gamma^{\bar{b}\bar{c}} \ , \ \bar{\gamma}\gamma^{\bar{a}} + \frac{i}{2}\epsilon^{abc}\gamma^{bc} \ . \tag{A.9}$$

The fourteen real combinations of these fourteen matrices generate G_2.

The $SU(3)$ subgroup is easily exhibited. First choose a basis for the 7 dimensional spinor subspace orthogonal to η:

$$\gamma^a\eta\ ,\ \gamma^{\bar{a}}\eta\ ,\ \bar{\gamma}\eta. \tag{A.10}$$

These give a basis for the **7** of G_2. Working in this basis, the real generators $\gamma^{a\bar{b}}+\gamma^{\bar{a}b}$, $i(\gamma^{a\bar{b}}-\gamma^{\bar{a}b})$, (with $a\neq b$) and $i(\gamma^{a\bar{a}}-\gamma^{b\bar{b}})$ correspond to matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda^T & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where λ are 3×3 generators of $SU(3)$. Thus these generators give the $SU(3)$ subgroup, and the decomposition $\mathbf{7}=\mathbf{3}+\bar{\mathbf{3}}+\mathbf{1}$ is manifest. One can also easily work out the explicit matrix representations of the remaining 6 generators (A.9) of G_2.

A.2. Invariants of G_2

As shown in table I, G_2 has a fully antisymmetric invariant $f^{\alpha\beta\gamma}$. this can easily be obtained from the spinor η:

$$f^{\alpha\beta\gamma}=\eta^T\Gamma^{\alpha\beta\gamma}\eta, \tag{A.11}$$

with normalization $\eta^T\eta=1$. Using the preceding construction of η, the $f^{\alpha\beta\gamma}$'s can be explicitly computed.

One cannot form lower G_2 invariants, such as $\eta^T\Gamma^\alpha\eta$ or $\eta^T\Gamma^{\alpha\beta}\eta$, because of the antisymmetry of $\Gamma^\alpha,\Gamma^{\alpha\beta}$. The invariants $\eta^T\Gamma^{\alpha\beta\gamma\delta}\eta$, $\cdots$, $\eta^T\Gamma^{\alpha_1\cdots\alpha_7}\eta$ are trivially duals of the lower invariants.

At first sight it would appear that there are many other invariants that can be constructed from the primitives $\delta^{\alpha\beta}$, $f^{\alpha\beta\gamma}$, and $\epsilon^{\alpha_1\cdots\alpha_7}$ by contracting products. However, the primitives satisfy a number of relations that restrict the number of possible invariants. A useful starting point is the Fierz identity

$$\eta^T\gamma^{\alpha\beta\gamma}\eta\,\eta^T\gamma^{\gamma\delta\epsilon}\eta=\frac{1}{8}\eta^T\gamma^{\alpha\beta\gamma}\gamma^{\gamma\delta\epsilon}\eta+\frac{1}{48}\eta^T\gamma^{\phi\eta\kappa}\eta\,\eta^T\gamma^{\alpha\beta\gamma}\gamma^{\phi\eta\kappa}\gamma^{\gamma\delta\epsilon}\eta. \tag{A.12}$$

From this one can show that the totally antisymmetrized product of two f's is equivalent to the dual of f,

$$f^{[\alpha\beta\gamma}f^{\gamma\delta\epsilon]}=\tilde{f}^{\alpha\beta\delta\epsilon}, \tag{A.13}$$

where on the left we antisymmetrize on $\alpha\beta\delta\epsilon$ and on the right $\tilde{f}$ is given by

$$\tilde{f}^{\alpha\beta\gamma\delta} = \frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta\epsilon\phi\eta} f^{\epsilon\phi\eta}. \tag{A.14}$$

Likewise, one can prove the identity [28]

$$f^{\alpha\beta\gamma} f^{\alpha\delta\epsilon} + f^{\alpha\delta\gamma} f^{\alpha\beta\epsilon} = 2\delta^{\beta\delta}\delta^{\gamma\epsilon} - \delta^{\gamma\delta}\delta^{\beta\epsilon} - \delta^{\beta\gamma}\delta^{\delta\epsilon} \tag{A.15}$$

which shows that the remaining components of the product of two f's are not independent. The triple identity

$$\begin{aligned} f^{\alpha\beta\gamma} f^{\gamma\delta\epsilon} f^{\epsilon\phi\eta} =& \delta^{\delta\alpha} f^{\beta\phi\eta} + \delta^{\phi\alpha} f^{\beta\eta\delta} + \delta^{\alpha\eta} f^{\beta\delta\phi} + \delta^{\delta\eta} f^{\alpha\beta\phi} \\ & - \delta^{\delta\beta} f^{\alpha\phi\eta} - \delta^{\beta\phi} f^{\alpha\eta\delta} - \delta^{\eta\beta} f^{\alpha\delta\phi} - \delta^{\delta\phi} f^{\alpha\beta\eta} \end{aligned} \tag{A.16}$$

also follows, and shows that any higher product is not independent.

Finally, an important relation comes from the observation that the fully antisymmetrized product $f^{[\alpha\beta\gamma}\tilde{f}^{\delta\epsilon\phi\eta]}$ must be proportional to $\epsilon^{\alpha\beta\gamma\delta\epsilon\phi\eta}$. Contracting both with $\epsilon^{\alpha\beta\gamma\delta\epsilon\phi\eta}$ and using

$$f^{\alpha\beta\gamma} f^{\alpha\beta\gamma} = 7 \cdot 6$$

(from (A.15)) shows that

$$\epsilon^{\alpha\beta\gamma\delta\epsilon\phi\eta} = 5\ f^{[\alpha\beta\gamma}\tilde{f}^{\delta\epsilon\phi\eta]}\ . \tag{A.17}$$

As a result, the invariants $\delta^{\alpha\beta}, f^{\alpha\beta\gamma}$ and $\tilde{f}^{\alpha\beta\gamma\delta}$ can be taken to be the only independent invariants. Other invariant tensors can always be reduced to products of these.

The identity (A.17) can be used to prove identities relating mesons and baryons. Substituting (A.17) into the relation[3]

$$3!\delta^{\alpha'\beta'\gamma'\delta'}_{\alpha\beta\gamma\delta} = \epsilon^{\alpha'\beta'\gamma'\delta'\mu\nu\lambda}\epsilon_{\alpha\beta\gamma\delta\mu\nu\lambda} \tag{A.18}$$

leads to the identity

$$\begin{aligned} 0 =& 7\delta^{\alpha'\beta'\gamma'\delta'}_{\alpha\beta\gamma\delta} + 3\tilde{f}_{\alpha\beta\gamma\delta}\tilde{f}^{\alpha'\beta'\gamma'\delta'} + 32\delta^{[\alpha'}_{[\alpha} f^{\beta'\gamma'\delta']} f_{\beta\gamma\delta]} \\ & - 16\delta^{[\alpha'}_{[\alpha}\tilde{f}^{\beta'\gamma'\delta']\epsilon}\tilde{f}_{\beta\gamma\delta]\epsilon} - 72\delta^{[\alpha}_{[\alpha}\delta^{\beta'}_{\beta} f^{\gamma'\delta']}{}_{,\gamma\delta]} \end{aligned} \tag{A.19}$$

where

$$f_{\alpha\beta,\gamma\delta} = f_{\alpha\beta\epsilon} f_{\gamma\delta\epsilon}\ .$$

[3] In subsequent formulas the height of indices is only significant for its convenience.

This can be simplified into a form more useful for proving proving the constraints in Section 3. Using two relations derived from the triple identity (by contracting it with an $f_{\mu\nu\lambda}$ on one and two indices)

$$\tilde{f}^{\beta'\gamma'\delta'\epsilon}\tilde{f}_{\beta\gamma\delta\epsilon} = 3(\delta^{[\beta'}_{[\beta} f^{\gamma'\delta']}{}_{,\gamma\delta]} - f^{[\beta'\gamma'}_{[\beta} f^{\delta']}_{\gamma\delta]})$$

and

$$f^{\gamma'\delta'}{}_{,\gamma\delta} = \tilde{f}^{\gamma'\delta'}_{\gamma\delta} + \delta^{\gamma'\delta'}_{\gamma\delta},$$

we have

$$\begin{aligned} 0 =& \tilde{f}_{\alpha\beta\gamma\delta}\tilde{f}^{\alpha'\beta'\gamma'\delta'} + \frac{32}{3}\delta^{[\alpha'}_{[\alpha} f^{\beta'\gamma'\delta']} f_{\beta\gamma\delta]} + 16\delta^{[\alpha'}_{[\alpha} f_{\beta}{}^{\beta'\gamma'} f^{\delta']}_{\gamma\delta]} \\ & -40\delta^{[\alpha'}_{[\alpha}\delta^{\beta'}_{\beta}\tilde{f}^{\gamma'\delta']}_{\gamma\delta]} - \delta^{\alpha'\beta'\gamma'\delta'}_{\alpha\beta\gamma\delta} \end{aligned} \tag{A.20}$$

This identity implies the constraints (3.19), (3.27) relating mesons and baryons.

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